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A modified inexact operator splitting method for monotone variational inequalities

Min Li · Abdellah Bnouhachem

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Abstract The Douglas–Peaceman–Rachford–Varga operator splitting methods (DPRV methods) are attractive methods for monotone variational inequalities. He et al. [Numer. Math. **94**, 715–737 (2003)] proposed an inexact self-adaptive operator splitting method based on DPRV. This paper relaxes the inexactness restriction further. And numerical experiments indicate the improvement of this relaxation.

Keywords Inexactness restriction · Operator splitting method · Variational inequalities

Mathematics Subject Classification (2000) 65K10 · 90C25 · 90C30

1 Introduction

Let Ω be a nonempty closed convex subset of R^n and F be a continuous monotone mapping from R^n into itself. The variational inequality problem is to determine a vector $u^* \in \Omega$ such that

$$\mathrm{VI}(\Omega, F) \qquad (u - u^*)^T F(u^*) \geq 0, \qquad \forall u \in \Omega. \tag{1}$$

$\mathrm{VI}(\Omega, F)$ includes nonlinear complementarity problems (when $\Omega = R^n_+$) and system of nonlinear equations (when $\Omega = R^n$), and thus it has many important applications, e.g., see [4,5].

It is well known that the problem $\mathrm{VI}(\Omega, F)$ is equivalent to the projection equation

$$u = P_\Omega[u - \beta F(u)],$$

M. Li (✉)
Department of Management Science and Engineering, School of Economics and Management, Southeast University, Nanjing 210096, China
e-mail: liminnju@yahoo.com

A. Bnouhachem
School of Management Science and Engineering, Nanjing University, Nanjing 210093, China

where β is a positive constant and $P_\Omega(\cdot)$ denotes the projection under the Euclidean norm of a point onto the set Ω, i.e.,

$$P_\Omega(v) = \text{argmin}\{\|v - u\| \mid u \in \Omega\}.$$

As a direct result, solving VI(Ω, F) is equivalent to finding a zero point of the continuous nonsmooth function

$$e(u, \beta) := u - P_\Omega[u - \beta F(u)]. \tag{2}$$

An advantageous method for solving variational inequality problems is the Douglas–Peaceman–Rachford–Varga operator splitting method (for short DPRV method, see p. 240 in [14]), which combines the Peaceman–Rachford algorithm [12] and the Douglas–Rachford algorithm [2]. For given $u^k \in R^n$, γ and β, let u_*^{k+1} be the solution of the following system of nonlinear smooth equations:

$$\text{compute } u \in R^n \text{ such that } \quad u + \beta F(u) - u^k - \beta F(u^k) + \gamma e(u^k, \beta) = 0. \tag{3}$$

The new iterate u^{k+1} of the exact version of DPRV method is taken by

$$u^{k+1} := u_*^{k+1}.$$

Some properties of such methods have been studied by Lions and Mercier [10], and Fukushima [6]. However sometimes it could be very expensive or even impossible to solve the system of nonlinear Eq. (3) exactly. In 2003, He et al. [9] proposed self-adaptive operator splitting methods to seek an inexact solution of (3) as the new iterate u^{k+1}, which is requested to satisfy the following condition:

$$\|u^{k+1} + \beta_k F(u^{k+1}) - u^k - \beta_k F(u^k) + \gamma e(u^k, \beta_k)\| \le \eta_k \|e(u^k, \beta_k)\|, \tag{4}$$

where $\{\eta_k\}$ is a nonnegative sequence with $\sum_{k=0}^{\infty} \eta_k^2 < +\infty$. This condition is still strict. Solving each subproblem usually requires numerous function values. In practice, these evaluations may be costly and time-consuming. So numerical methods which could effectively reduce the number of employing function values are highly desired.

In this paper, we study a modified inexact operator splitting method which significantly relaxes the inexactness restriction to

$$\|u^{k+1} + \beta_k F(u^{k+1}) - u^k - \beta_k F(u^k) + \gamma e(u^k, \beta_k)\| \le \nu_k \|e(u^k, \beta_k) - e(u^{k+1}, \beta_k)\|,$$

$$\text{with} \qquad \sup \nu_k = \nu < \frac{2-\gamma}{2} \qquad \text{and} \qquad \gamma \in (0, 2).$$

To illustrate the superiority of this improvement, some numerical experiments will be presented in Sect. 4.2.

Throughout this paper we assume that the operator F is monotone and that the solution set of VI(Ω, F), denoted by Ω^*, is nonempty and contains at least one finite element. We use u^* to denote any finite point in Ω^*.

2 Preliminaries

This section states some preliminaries that are useful in the convergence analysis of this paper. First, projection on a closed convex set plays an important role in our analysis. A well known property of the projection mapping is

$$(v - P_\Omega(v))^T (u - P_\Omega(v)) \le 0, \qquad \forall v \in R^n, u \in \Omega. \tag{5}$$

By setting $v = \bar{u} - \beta F(\bar{u})$, $u = P_\Omega[\tilde{u} - \beta F(\tilde{u})]$ and $v = \tilde{u} - \beta F(\tilde{u})$, $u = P_\Omega[\bar{u} - \beta F(\bar{u})]$ in (5) respectively, we obtain

$$\{\bar{u} - \beta F(\bar{u}) - P_\Omega[\bar{u} - \beta F(\bar{u})]\}^T \{P_\Omega[\bar{u} - \beta F(\bar{u})] - P_\Omega[\tilde{u} - \beta F(\tilde{u})]\} \geq 0 \tag{6}$$

and

$$\{P_\Omega[\tilde{u} - \beta F(\tilde{u})] - \tilde{u} + \beta F(\tilde{u})\}^T \{P_\Omega[\bar{u} - \beta F(\bar{u})] - P_\Omega[\tilde{u} - \beta F(\tilde{u})]\} \geq 0. \tag{7}$$

Adding (6) and (7) we get

$$\{[e(\bar{u}, \beta) - e(\tilde{u}, \beta)] + \beta[F(\tilde{u}) - F(\bar{u})]\}^T \{[e(\tilde{u}, \beta) - e(\bar{u}, \beta)] + (\bar{u} - \tilde{u})\} \geq 0. \tag{8}$$

Combining this inequality with the monotonicity of F, we get the following lemma immediately.

Lemma 1 *For all $\bar{u}, \tilde{u} \in R^n$ and $\beta > 0$ we have*

$$\begin{aligned} &\{(\bar{u} - \tilde{u}) + \beta[F(\bar{u}) - F(\tilde{u})]\}^T [e(\bar{u}, \beta) - e(\tilde{u}, \beta)] \\ &\quad \geq \|e(\tilde{u}, \beta) - e(\bar{u}, \beta)\|^2. \end{aligned} \tag{9}$$

For given $u \in R^n$, the magnitude $\|e(u, \beta)\|$ is dependent on β. For simplicity, we write $e(u)$ to represent $e(u, 1)$. The following properties of $\|e(u, \beta)\|$ are needed in the convergence analysis.

Lemma 2 *For any $u \in R^n$ and $\tilde{\beta} \geq \beta > 0$, we have*

$$\|e(u, \tilde{\beta})\| \geq \|e(u, \beta)\| \tag{10}$$

and

$$\frac{\|e(u, \tilde{\beta})\|}{\tilde{\beta}} \leq \frac{\|e(u, \beta)\|}{\beta}. \tag{11}$$

Proof See [15] for a simple proof. □

Recall that $u \in \Omega^*$ is equivalent to $e(u) = 0$ (see p. 267 in [1]), we take $\|e(u)\| \leq \varepsilon$ as the stopping criterion. Convergence means that for any given $\varepsilon > 0$ the proposed method will find a $u^k \in \Omega$ in finite iterations such that $\|e(u^k)\| \leq \varepsilon$. The following theorem can be derived from Theorem 2 of [9]. For completeness, a proof is provided.

Theorem 1 *Let $c > 0$ be a constant, $\{\beta_k\} \subset [B_L, +\infty)$ with $B_L > 0$ and $\{\xi_k\}$ be nonnegative with $\sum_{k=0}^{\infty} \xi_k < +\infty$. If there is a $k_0 > 0$ such that for all $k \geq k_0$ and for any u^* (finite) $\in \Omega^*$, the sequence $\{u^k\}$ satisfies*

$$\begin{aligned} &\|(u^{k+1} - u^*) + \beta_{k+1}[F(u^{k+1}) - F(u^*)]\|^2 \\ &\quad \leq (1 + \xi_k)\|(u^k - u^*) + \beta_k[F(u^k) - F(u^*)]\|^2 - c\|e(u^k, \beta_k)\|^2, \end{aligned} \tag{12}$$

then the method which generated $\{u^k\}$ is convergent.

Proof From $\sum_{k=0}^{\infty} \xi_k < +\infty$ it follows that $\prod_{i=0}^{\infty}(1 + \xi_i) < +\infty$. We denote

$$C_s := \sum_{i=0}^{\infty} \xi_i \quad \text{and} \quad C_p := \prod_{i=0}^{\infty}(1 + \xi_i). \tag{13}$$

Let $\tilde{u}$ (finite) $\in \Omega^*$. From (12) we get

$$\begin{aligned}
&\|(u^{k+1} - \tilde{u}) + \beta_{k+1}[F(u^{k+1}) - F(\tilde{u})]\|^2 \\
&\leq \prod_{i=k_0}^{k} (1 + \xi_i)\|(u^{k_0} - \tilde{u}) + \beta_{k_0}[F(u^{k_0}) - F(\tilde{u})]\|^2 \\
&\leq C_p\|(u^{k_0} - \tilde{u}) + \beta_{k_0}[F(u^{k_0}) - F(\tilde{u})]\|^2, \qquad \forall k \geq k_0.
\end{aligned}$$

Therefore, there exists a constant $C > 0$ such that

$$\|(u^k - \tilde{u}) + \beta_k[F(u^k) - F(\tilde{u})]\|^2 \leq C, \qquad \forall k \geq 0. \tag{14}$$

From (14) and the monotonicity of F, it is easy to verify that the sequence $\{u^k\}$ is bounded. Combining (12) and (14), we have

$$\begin{aligned}
c \sum_{k=k_0}^{\infty} \|e(u^k, \beta_k)\|^2 \leq \|(u^{k_0} - \tilde{u}) + \beta_{k_0}[F(u^{k_0}) - F(\tilde{u})]\|^2 \\
+ \sum_{k=k_0}^{\infty} \xi_k\|(u^k - \tilde{u}) + \beta_k[F(u^k) - F(\tilde{u})]\|^2 \\
\leq C + C\sum_{k=k_0}^{\infty} \xi_k \\
\leq (1 + C_s)C
\end{aligned} \tag{15}$$

and it follows from Lemma 2 that

$$\lim_{k\to\infty} e(u^k, B_L) = 0.$$

The proof is complete. □

For the convergence proof, as in [3] and [8], we need the following analytical results.

Lemma 3 *Let $\{a_k\}_{k=0}^{\infty}$ be a positive series and $a_k \in (0, 1)$ for all k. If $\prod_{k=0}^{\infty}(1 - a_k) > 0$, then*

1. $\sum_{k=0}^{\infty} a_k < +\infty$ *and thus* $\lim_{k\to\infty} a_k = 0$;
2. $\prod_{k=0}^{\infty}(1 + ta_k) < \infty$ *for any* $t > 0$.

Proof The proof follows from elementary mathematical analysis and thus is omitted. □

3 The modified method and its convergence

In this section, we construct a modified method for monotone variational inequalities and then prove its convergence. First, we state the framework of this new method.

For given $\gamma \in (0, 2)$, $\beta_k > 0$ and $u^k \in R^n$, if $u^k \notin \Omega^*$ find an approximate solution of (3), i.e., find u^{k+1} in the sense that

$$\Theta_k(u^{k+1}) := u^{k+1} + \beta_k F(u^{k+1}) - u^k - \beta_k F(u^k) + \gamma e(u^k, \beta_k) \tag{16}$$

under the following inexactness restriction:

$$\|\Theta_k(u^{k+1})\| \leq \nu_k\|e(u^k, \beta_k) - e(u^{k+1}, \beta_k)\| \quad \text{with} \quad \sup \nu_k = \nu < \frac{2-\gamma}{2}. \tag{17}$$

Since the proposed method is a modified form of operator splitting methods and u^{k+1} is the inexact zero point of

$$\Theta_k(u) := u + \beta_k F(u) - u^k - \beta_k F(u^k) + \gamma e(u^k, \beta_k),$$

we refer the above method as *a modified inexact operator splitting method*. For convenience, we assume that $\beta_k \equiv \beta$ in the following analysis.

The main property of the sequence generated by the proposed method is listed in the following theorem. Instead of forcing to reduce the unknown distance value $\|(u - u^*) + \beta[F(u) - F(u^*)]\|^2$ in each iteration, it will show that the sequence $\{\|e(u^k, \beta)\|\}$ is monotone non-increasing.

Theorem 2 *Let* $\{u^k\}$ *be the sequence generated by* (16)–(17). *Then we have*

$$\|e(u^{k+1}, \beta)\|^2 \le \|e(u^k, \beta)\|^2 - c_1 \|e(u^k, \beta) - e(u^{k+1}, \beta)\|^2 \quad \text{with} \quad c_1 = \frac{2 - (\gamma + 2\nu)}{\gamma}. \tag{18}$$

Proof By a manipulation, we have

$$\|e(u^k, \beta)\|^2 = \|e(u^{k+1}, \beta)\|^2 - \|e(u^k, \beta) - e(u^{k+1}, \beta)\|^2 + 2e(u^k, \beta)^T [e(u^k, \beta) - e(u^{k+1}, \beta)]. \tag{19}$$

Setting $\bar{u} = u^{k+1}$ and $\tilde{u} = u^k$ in (9), we have

$$\{(u^{k+1} - u^k) + \beta[F(u^{k+1}) - F(u^k)]\}^T [e(u^{k+1}, \beta) - e(u^k, \beta)] \ge \|e(u^k, \beta) - e(u^{k+1}, \beta)\|^2. \tag{20}$$

Note that (see (16))

$$u^{k+1} - u^k + \beta[F(u^{k+1}) - F(u^k)] = \Theta_k(u^{k+1}) - \gamma e(u^k, \beta).$$

Hence, it follows from (20) that

$$\gamma e(u^k, \beta)^T [e(u^k, \beta) - e(u^{k+1}, \beta)] \ge \|e(u^k, \beta) - e(u^{k+1}, \beta)\|^2 + \Theta_k(u^{k+1})^T [e(u^k, \beta) - e(u^{k+1}, \beta)]. \tag{21}$$

Substituting (21) into (19), we get

$$\|e(u^k, \beta)\|^2 - \|e(u^{k+1}, \beta)\|^2 + \|e(u^k, \beta) - e(u^{k+1}, \beta)\|^2 \ge \frac{2}{\gamma}\left(\|e(u^k, \beta) - e(u^{k+1}, \beta)\|^2 - \|\Theta_k(u^{k+1})\| \cdot \|e(u^k, \beta) - e(u^{k+1}, \beta)\|\right) \tag{22}$$

The assertion of this theorem follows from (17) and (22) immediately. □

Let $\{u^k\}$ be the sequence generated by the proposed method. Since $\|e(u^k, \beta)\| \ne 0$ (otherwise u^k is a solution), we can define

$$\eta_k = \frac{\nu_k \|e(u^k, \beta) - e(u^{k+1}, \beta)\|}{\|e(u^k, \beta)\|}. \tag{23}$$

It follows from this and (17) that

$$\|\Theta_k(u^{k+1})\| \le \eta_k \|e(u^k, \beta)\|. \tag{24}$$

The following theorem concerns a contractive-like property which is important to the convergence analysis.

Theorem 3 *Let $\{u^k\}$ be the sequence generated by the modified inexact operator splitting method. If the nonnegative scalar sequence $\{\eta_k\}$ defined by (23) satisfies $\sum_{k=0}^{\infty}\eta_k^2 < +\infty$, then there exists a $k_B > 0$ such that for any $k \geq k_B$ and any u^* (finite) $\in \Omega^*$, we have*

$$\begin{aligned}&\|(u^{k+1}-u^*)+\beta[F(u^{k+1})-F(u^*)]\|^2\\&\quad\leq(1+\xi_k)\|(u^k-u^*)+\beta[F(u^k)-F(u^*)]\|^2-c_2\|e(u^k,\beta)\|^2,\end{aligned}\tag{25}$$

where

$$\xi_k=\frac{4\eta_k^2}{\gamma(2-\gamma)}\qquad\text{and}\qquad c_2=\frac{\gamma(2-\gamma)}{2}.\tag{26}$$

Proof The proof is similar to the one for Theorem 4 of [8]. By setting $\bar{u}=u^k$ and $\tilde{u}=u^*$ in (9) and using $e(u^*,\beta)=0$, we have

$$\{(u^k-u^*)+\beta[F(u^k)-F(u^*)]\}^T e(u^k,\beta)\geq\|e(u^k,\beta)\|^2$$

It follows from this and (16) that

$$\begin{aligned}&\|(u^{k+1}-u^*)+\beta[F(u^{k+1})-F(u^*)]\|^2\\&\quad=\|(u^k-u^*)+\beta[F(u^k)-F(u^*)]-(\gamma e(u^k,\beta)-\Theta_k(u^{k+1}))\|^2\\&\quad\leq\|(u^k-u^*)+\beta[F(u^k)-F(u^*)]\|^2-2\gamma\|e(u^k,\beta)\|^2\\&\qquad+2\{(u^k-u^*)+\beta[F(u^k)-F(u^*)]\}^T\Theta_k(u^{k+1})+\|\gamma e(u^k,\beta)-\Theta_k(u^{k+1})\|^2.\end{aligned}\tag{27}$$

Using Cauchy-Schwarz inequality and (24), we have

$$\begin{aligned}&2\{(u^k-u^*)+\beta[F(u^k)-F(u^*)]\}^T\Theta_k(u^{k+1})\\&\quad\leq\frac{4\eta_k^2}{\gamma(2-\gamma)}\|(u^k-u^*)+\beta[F(u^k)-F(u^*)]\|^2+\frac{\gamma(2-\gamma)}{4\eta_k^2}\|\Theta_k(u^{k+1})\|^2\\&\quad\leq\frac{4\eta_k^2}{\gamma(2-\gamma)}\|(u^k-u^*)+\beta[F(u^k)-F(u^*)]\|^2+\frac{\gamma(2-\gamma)}{4}\|e(u^k,\beta)\|^2.\end{aligned}\tag{28}$$

From (24) and $\sum_{i=0}^{\infty}\eta_i^2 < +\infty$, it is easy to show that there exists a $k_B \geq 0$ such that for any $k \geq k_B$

$$\|\gamma e(u^k,\beta)-\Theta_k(u^{k+1})\|^2\leq\gamma^2\|e(u^k,\beta)\|^2+\frac{\gamma(2-\gamma)}{4}\|e(u^k,\beta)\|^2.\tag{29}$$

Substituting (28) and (29) into (27), we complete the proof. □

Now we are in the stage to prove the convergence of the proposed method.

Corollary 1 *The modified inexact operator splitting method for monotone variational inequalities is convergent.*

Proof The proof is indirect. Since the sequence $\{\|e(u^k,\beta)\|\}$ generated by the modified inexact self-adaptive projection method is monotone non-increasing, if the method is not convergent, we have

$$\lim_{k\to\infty}\|e(u^k,\beta)\|=\omega>0.\tag{30}$$

It follows from Theorem 2 and (23) that

$$\frac{\|e(u^{k+1},\beta)\|^2}{\|e(u^k,\beta)\|^2}\leq 1-c_1\frac{\|e(u^k,\beta)-e(u^{k+1},\beta)\|^2}{\|e(u^k,\beta)\|^2}\leq 1-(\frac{c_1}{\nu^2})\eta_k^2\tag{31}$$

and consequently

$$\prod_{k=0}^{\infty} \frac{\|e(u^{k+1}, \beta)\|^2}{\|e(u^k, \beta)\|^2} \le \prod_{k=0}^{\infty} \left(1 - (\frac{c_1}{\nu^2})\eta_k^2\right). \tag{32}$$

From (30) and (32) we have

$$\prod_{k=0}^{\infty}\left(1 - (\frac{c_1}{\nu^2})\eta_k^2\right) \ge \frac{\omega^2}{\|e(u^1, \beta)\|^2} > 0. \tag{33}$$

Then, it follows from Lemma 3 and $0 < (c_1/\nu^2)\eta_k^2 < 1$ (see (31)) that

$$\sum_{k=0}^{\infty} \eta_k^2 < +\infty, \quad \lim_{k\to\infty} \eta_k^2 = 0 \quad \text{and} \quad \prod_{k=0}^{\infty}(1 + t\eta_k^2) < \infty, \quad \forall\, t > 0. \tag{34}$$

From this and (26), we have $\sum_{k=0}^{\infty} \xi_k < +\infty$. Then it follows directly from (25) and Theorem 1 that

$$\lim_{k\to\infty} \|e(u^k, \beta)\| = 0,$$

which contradicts (30) and the theorem is proved. □

4 Implementation details and numerical experiments

In this section, we present implementation details and some numerical results for the proposed inexact operator splitting method. Our main interest is to demonstrate the computational superiority of new relaxed inexactness restriction over the original one in He's DPRV method [9]. The subproblem of the new method is easier to solve, which means that the number of calculations of $F(u)$ is reduced effectively.

4.1 Implementation details

From numerical point of view, $\beta\|F(u^{k+1}) - F(u^k)\| \approx \|u^{k+1} - u^k\|$ is favorable for fast convergence. According to our numerical experiences, we suggest to adjust β_k such that:

$$\frac{1}{(1+s)} \le \frac{\|\beta_k[F(u^{k+1}) - F(u^k)]\|}{\|u^{k+1} - u^k\|} \le 1 + s,$$

where s is a given positive constant. In the new method, we set $s = 2$.

By considering the above improvements, we obtain the following modified inexact operator splitting method.

Algorithm For given $\gamma \in (0, 2)$, $\beta_0 > 0$ and $u^0 \in R^n$, set $k = 0$. The sequence $\{u^k\}$ is generated by the iterative schemes:

Step 1. Find an approximate solution of (3), i.e., find u^{k+1} in the sense that

$$\Theta_k(u^{k+1}) := u^{k+1} + \beta_k F(u^{k+1}) - u^k - \beta_k F(u^k) + \gamma e(u^k, \beta_k)$$

under the following inexactness restriction:

$$\|\Theta_k(u^{k+1})\| \le \nu_k \|e(u^k, \beta_k) - e(u^{k+1}, \beta_k)\| \quad \text{with} \quad \sup \nu_k = \nu < \frac{2-\gamma}{2}.$$

Step 2. If the given stopping criterion is satisfied, then stop. Otherwise set

$$\omega_k = \frac{\|\beta_k[F(u^{k+1}) - F(u^k)]\|}{\|u^{k+1} - u^k\|}, \tag{35}$$

and adjust the scaling parameter β_k

$$\beta_k = \begin{cases} 3 * \beta_k & \text{if } \omega_k < 0.3, \\ \beta_k/3 & \text{if } \omega_k > 3, \\ \beta_k & \text{otherwise.} \end{cases} \tag{36}$$

Set $k := k + 1$ and go to Step 1.

Remark 1 The strategy of adjusting the value of β_k only needs to be applied for finite number of steps, which does not affect the convergence of the proposed method.

To compare with He's DPRV method [9], we use the same gradient method [8] to solve the system of nonlinear equations

$$\textbf{(SNE)} \qquad u + \beta_k F(u) = u^k + \beta_k F(u^k) - \gamma e(u^k, \beta_k) \tag{37}$$

inexactly (with the inexactness restriction (17)). Assume that F is Lipschitz continuous (say with constant L). Note that problem (37) is a system of nonlinear equations of type $\Theta_k(z) = 0$ (see (16)) and the mapping Θ_k is Lipschitz continuous

$$\|\Theta_k(x) - \Theta_k(y)\| \le (1 + \beta_k L)\|x - y\|, \qquad \forall x, y \in R^n$$

and strongly monotone with a constant modulus 1

$$\|x - y\|^2 \le (x - y)^T[\Theta_k(x) - \Theta_k(y)], \qquad \forall x, y \in R^n$$

For completeness, we list the implementation details of Step 1 in the following.

A gradient type method

Step 1.0. *Given* $\delta \in (0, 1)$, $\mu \in [0.5, 1)$, $\varepsilon > 0$, $\alpha > 0$ *and* $z^0 \in R^n$, *set* $i = 0$.
Step 1.1. *If* $\|\Theta_k(z^i)\| \le \nu_k\|e(u^k, \beta_k) - e(z^i, \beta_k)\|$, *set* $u^{k+1} = z^i$ *and then stop. Otherwise, go to Step 1.2.*
Step 1.2. *Find the smallest nonnegative integer* l_i, *such that* $\alpha_i = \mu^{l_i}\alpha$ *and*

$$z^{i+1} = z^i - \alpha_i \Theta_k(z^i) \tag{38}$$

satisfies

$$r_i := \frac{\alpha_i\|\Theta_k(z^i) - \Theta_k(z^{i+1})\|^2}{(z^i - z^{i+1})^T(\Theta_k(z^i) - \Theta_k(z^{i+1}))} \le 2 - \delta. \tag{39}$$

Step 1.3. *Adjust* α_i *for the next step to avoid too small improvement*

$$\alpha_i = \begin{cases} \alpha_i * 1.5 & \textit{if } r_i \le 0.5, \\ \alpha_i & \textit{otherwise.} \end{cases}$$

Set $i = i + 1$, *and go to Step 1.1.*

Table 1 Numerical results for NCP with different n

n	He's DPRV method		The modified method	
	No. of It. k	No. of Nf	No. of It. k	No. of Nf
20	61	1673	56	929
100	68	1900	66	974
200	98	2944	102	1751
500	101	3037	96	2168
1000	110	3378	111	2512
2000	116	3601	126	2695

4.2 Numerical experiments

We consider the nonlinear complementarity problems (for short NCP): Find $u \in R^n$ such that

$$u \geq 0, \qquad F(u) \geq 0, \qquad u^T F(u) = 0. \tag{40}$$

In our test problem we take

$$F(u) = D(u) + Mu + q, \tag{41}$$

where $D(u)$ and $Mu + q$ are the nonlinear part and the linear part of $F(u)$, respectively. We form the linear part $Mu + q$ similarly as in [7].[1] The matrix $M = A^T A + B$, where A is an $n \times n$ matrix whose entries are randomly generated in the interval $(-5, +5)$ and a skew-symmetric matrix B is generated in the same way. The vector q is generated from a uniform distribution in the interval $(-500, 0)$. In $D(u)$, the nonlinear part of $F(u)$, the components are $D_j(u) = a_j * \arctan(u_j)$ and a_j is a random variable in (0, 1). A similar type of the problem was tested in [11] and [13].[2] We use the proposed method with the strategy of adjusting β_k to solve this set of problems. In each iteration the system of nonlinear equations (SNE) was solved inexactly under the inexactness restriction (17) with $\nu_k \equiv 0.2$. For comparison purpose, we also coded He's DPRV method (the same method with the inexactness restriction (4)). In He's DPRV method, we take $\eta_k = 1/(k^2 + 1)$.

All codes were written in Matlab and run on an IBM X31 Notebook personal computer. In all test examples we take $\beta_0 = 0.1$ and $\gamma = 1.5$. The iterations begin with $u^0 = (0, \ldots, 0) \in R^n$ and stop as soon as $\|e(u^k)\|_\infty \leq 10^{-7}$. We list the numbers of iterations and the numbers of mapping F evaluations (Nf) in Table 1 for the sake of comparison.

From the numerical results, we find that the inexactness restriction of the modified method is efficiently relaxed almost without increasing the iteration numbers. Note that the computational costs in each iteration of these methods depend on the numbers of mapping F evaluations greatly. For such constructed problems, we can observe from the above table that

$$\frac{\text{total computational load of the modified method}}{\text{total computational load of He's DPRV method}} < 0.75.$$

In addition, for a set of similar problems, it seems that these iteration numbers are not very sensitive to the problem size.

[1] In the paper by Harker and Pang [7], the matrix $M = A^T A + B + D$, where A and B are the same matrices as here, and D is a diagonal matrix with uniformly distributed random variable $d_{jj} \in (0, 0.3)$.

[2] In [11] and [13], the components of nonlinear mapping $D(u)$ are $D_j(u) = \text{constant} * \arctan(u_j)$.

5 Conclusions

In this paper, we propose a modified inexact operator splitting method for monotone variational inequalities. The method possesses stronger efficiency, since its inexactness restriction is much relaxed compared to that of He's DPRV method. The preliminary numerical tests show that the proposed method is attractive in practice.

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